4. Reproducibility of data was very much poorer in the 24-in. column, especially at the lower velocities.

No entirely satisfactory explanation can be offered to account for all these somewhat surprising results. The author expected larger bubbles in the large column and suspects that large bubbles were actually present but that only occasionally would the bubble paths contact the probe. One reason for believing this is the generally poorer reproducibility of the 24-in. data in comparison with the 4-in. data. Figure 9 shows a good example of this phenomenon. For the first part of the trace practically no bubbles were evident. This was followed immediately by a period in which many large bubbles contacted the probe. Since the probe was located in the horizontal center of the bed, it would have been possible for the bubbles periodically to channel around the probe through other vertical paths. Another reason for suspecting the presence of bubbles in the 24-in. column (but generally undetected by the capacitance probe) is the difference between the over-all density (measured by bed volume and powder weight) and local density at the probe. This difference is in the direction one would expect for such an occurrence. An example of the difference in density is as follows:

Run 24-2	
Powder size:	−60 mesh
Probe height:	48 in.
Superficial gas velocity:	0.5 ft./sec.
Over-all density:	47.5 lb./cu. ft.
Local density at probe:	59.1 lb./cu. ft.

These are believed to be real differences in density. The presence of large bubbles or bubble paths in this column was confirmed subsequently by the helium tracer technique, which showed gross gas bypassing. Because of this nonrepresentative sampling in the center of the bed only, the results obtained in the 24-in. column and reported here should not be used as a true indication of the effect of the primary variables throughout the entire bed.

CONCLUSIONS

4-inch Column

1. Over the ranges studied, all operating variables except bed height had a measurable influence on the uniformity of density throughout the bed.

2. Gas velocity through the powder bed had by far the greatest effect on uniformity, with uniformity generally decreasing with an increase in gas velocity. This was interpreted to mean that most gas introduced in excess of the rate for incipient fluidization passes through the bed in the form of various-sized bubbles.

3. Better uniformity at the lower bed level indicates that bubbles grow in size as they proceed up the column.

4. Bubble growth is accelerated by a coarser powder.

5. Fine powder promotes the formation of gas channels or bubble paths at low gas velocities in the lower section of the powder bed.

6. Except in the vicinity immediately above it the type of distributor has relatively little effect on uniformity.

Twenty-four-inch Column

1. The more uniform density (in comparison with that of the 4-in.) found in the horizontal center of the 24-in. column is not believed to be representative of the entire bed.

2. Gas channels or distinct bubble paths are believed to be present in the column.

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Viscous Flow Relative to Arrays of Cylinders

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The free-surface model, successfully employed to predict sedimentation, resistance to flow, and viscosity in assemblages of spherical particles, has been extended to the case of flow relative to cylinders. It is shown to be in good agreement with existing data on beds of fibers of various types and flow through bundles of heat-exchanger tubes for cases where it can reasonably be expected to apply. Close agreement in the dilute range with the only theoretical treatment for flow parallel to a square array of cylinders provides interesting validation of the model.

The steady slow motion of fluids relative to assemblages of spherical particles has been previously treated mathematically by the use of modification of the unit-cell technique. Results have been reported for sedimentation and resistance to flow (*11*) as well as for viscosity of suspensions (*10*). The present development extends this theory to the case of flow relative to groups of cylindrical objects.

The derivations are developed on the basis that two concentric cylinders can serve as the model for fluid moving through an assemblage of cylinders. The inner cylinder consists of one of the rods in the assemblage and the outer cylinder of a fluid envelope with a free surface. The relative volume of fluid to solid in the cell model is taken to be the same as the relative volume of fluid to solid in the assemblage of cylinders. In effect one assumes that at a distance from the disturbance to fluid motion caused by a cylinder the velocity of flow will not be greatly affected by the exact shape of the outside boundary. The important consideration is that the appropriate boundary condition of no slippage along the walls of the fluid envelope be maintained. The situation is easily visualized in Figure 1, which shows the unit cell in a square array as compared with the model assumed for axial flow. The cross-hatched area occupied by fluid is the same for both the array and model. The dotted line indicates the outside boundary of the fluid envelope at which a condition of no friction is maintained. Employment of appropriate boundary conditions en-

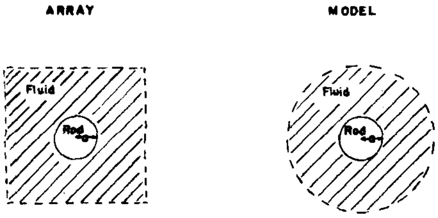

Fig. 1. Free-surface model for axial flow.

ables closed solutions to be obtained for the Navier-Stokes equations by the use of this model.

Two basic cases are considered; in the first case flow is assumed to be parallel to the axis of the cylinder, and in the second, flow is at right angles to the cylinder axis. These results are directly applicable to the cases where fluid is flowing parallel or perpendicular to a bank of cylinders. In the case of random assemblages, where the cylinders are not parallel to each other, it is necessary to employ a weighted average. In the case of flow perpendicular to cylinders the model does not distinguish between crossed or parallel arrangement of the cylinders, and in such random assemblages it is necessary in averaging to give twice the weight to the correlation for flow perpendicular to cylinders as for flow parallel to cylinders.

Thus

$$u = -\frac{1}{4\mu}\frac{dp}{dx} \cdot \left[(a^2 - r^2) + 2b^2 \ln \frac{r}{a}\right] \tag{5}$$

The flow rate through the entire annulus of fluid between $r = a$ and $r = b$ will be

$$Q = 2\pi \int_a^b ur\,dr = \frac{-\pi}{8\mu}\frac{dp}{dx} \cdot \left[4a^2b^2 - a^4 - 3b^4 + 4b^4 \ln \frac{b}{a}\right] \tag{6}$$

If $u_{avg} = Q/\text{area}$ and Darcy's equation for flow through a porous medium is written $u_{avg} = -(K/\mu)\,(dp)/(dx)$,

$$K = \frac{1}{8b^2}\left(4a^2b^2 - a^4 - 3b^4 + 4b^4 \ln \frac{b}{a}\right) \tag{7}$$

The well-known Carman-Kozeny (*3*) equation, derived on the basis of semi-empirical reasoning, also gives an expression for the Darcy constant

$$K = \frac{\epsilon m^2}{k} \tag{8}$$

In the present case $m = (b^2 - a^2)/2a$, and thus the Kozeny constant becomes

$$k = \frac{2\epsilon^3}{[1 - \epsilon]\left[2 \ln\left(\frac{1}{1 - \epsilon}\right) - 3 + 4(1 - \epsilon) - (1 - \epsilon)^2\right]} \tag{9}$$

FLOW PARALLEL TO CYLINDERS

The basic differential equation to be solved is

$$\frac{d}{r\,dr}\, r\frac{du}{dr} = \frac{1}{\mu}\frac{dp}{dx} \tag{1}$$

The solution to this equation for constant dp/dx is

$$u = \frac{1}{4\mu}\frac{dp}{dx} r^2 + A \ln r + B \tag{2}$$

In this case one assumes that fluid is moving through the annular space between the cylinder of radius a and the fluid envelope of radius b. (The same result could be obtained if the over-all fluid motion were assumed to be zero and the inner cylinder to be in motion.) These boundary conditions give

$$u = 0 \quad \text{at} \quad r = a \qquad \frac{du}{dr} = 0 \quad \text{at} \quad r = b \tag{3}$$

whence by substitution in (2),

$$A = -\frac{1}{2\mu}\frac{dp}{dx} b^2 \qquad B = \frac{1}{4\mu}\frac{dp}{dx}(2b^2 \ln a - a^2) \tag{4}$$

It should be noted that in this case the solution obtained applies to the complete Navier-Stokes equations.* The inertia terms vanish because there is no change in cross section along the direction of flow, hence no velocity change in the x direction.

FLOW PERPENDICULAR TO CYLINDERS

The inertia terms are omitted from the Navier-Stokes equations to obtain the creeping-motion equations. Expressed in cylindrical coordinates and for two dimensions these are

$$\frac{\partial p}{\partial r} = \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2\partial v_\theta}{r^2\partial\theta}\right)$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right)$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = 0 \tag{10}$$

It is convenient to employ the stream function defined by the relations

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = -\frac{\partial\psi}{\partial r} \tag{11}$$

*Sparrow and Loeffler (*12a*) present an analytical solution for the longitudinal laminar-flow between cylinders arranged in triangular or square array. For large spacings the result is equivalent to Equation (9) above. A similar result for square spacing was also obtained by L. S. Leibenson (*11a*).

Equations (10) then assume the form of the biharmonic equation

$$\nabla^4\psi = 0 \tag{12}$$

A suitable general solution of Equation (12) is

$$\psi = \sin\theta\left[\tfrac{1}{8}Cr^3 + \tfrac{1}{2}Dr(\ln r - \tfrac{1}{2}) + Er + \frac{F}{r}\right] \tag{13}$$

In this case a solid cylinder of radius a is assumed to be moving perpendicular to its axis in a fluid cell of radius b. (The same result would be obtained if one assumed that fluid is moving perpendicular to a stationary cylinder of radius a.) There is assumed to be no shearing stress on the outside cylindrical fluid surface.

Under these assumptions the boundary conditions are

$$\left.\begin{aligned} u &= U \\ v_r &= U\cos\theta \\ v_\theta &= -U\sin\theta \end{aligned}\right\} \text{at} \quad r = a \qquad \left.\begin{aligned} v_r &= 0 \\ \frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r} &= 0 \end{aligned}\right\} \text{at } r = b \tag{14}$$

These conditions provide four simultaneous equations to establish the values of the four constants. The drag on the solid cylinder $r = a$ requires the evaluation of only one constant, since (12)

$$W = 2\pi\mu D \tag{15}$$

The constant D is found to be

$$D = \frac{-2U}{\ln\frac{b}{a} + \frac{a^4}{b^4 + a^4} - \frac{1}{2}} \tag{16}$$

and thus

$$W = \frac{4\pi\mu U}{\ln\frac{b}{a} - \frac{1}{2}\left(\frac{b^4 - a^4}{b^4 + a^4}\right)} \tag{17}$$

If the cylinder remains stationary and fluid passes it, the force associated with a single cylindrical cell may be equated to the pressure gradient

$$\frac{W}{\pi b^2} = \frac{dp}{dx} \tag{18}$$

As previously stated, the Darcy equation may be written $u_{avg} = U = -(K/\mu)(dp/dx)$, and the Darcy constant is then found to be

$$K = \frac{b^2}{4}\left[\ln\frac{b}{a} - \frac{1}{2}\left(\frac{b^4 - a^4}{b^4 + a^4}\right)\right] \tag{19}$$

Similarly on the basis of the Carman-Kozeny theory $K = (\epsilon m^2/k)$, and the Kozeny constant becomes

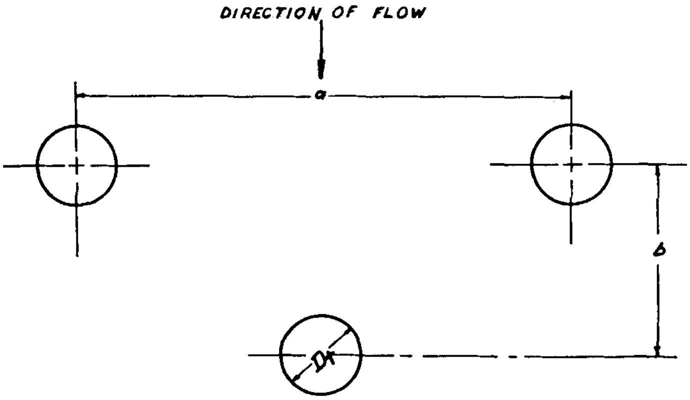

Fig. 2. Tube-layout-definition sketch.

$$k = \frac{2\epsilon^3}{[1-\epsilon]\left\{\ln\left(\frac{1}{1-\epsilon}\right)\left[\frac{1-(1-\epsilon)^2}{1+(1-\epsilon)^2}\right]\right\}} \qquad (20)$$

RESULTS AND DISCUSSION

Table 1 gives a comparison of values of the Kozeny constant computed from the above equations and the theoretical value for an assemblage of spherical particles (*11*) based on the free-surface model. Although the Kozeny constant for spheres lies between those for parallel and perpendicular flow with cylinders for fractional void volumes between 0.4 and 0.8, at higher values of voidage the Kozeny constant for spheres is higher than for both types of cylindrical assemblage. The model does not appear to be applicable to fractional void volumes below about 0.4 to 0.5.

Carman's book (*3*) gives a comprehensive survey of the available data on fibers for testing the Carman-Kozeny equation. He points out that elongated shapes such as fibers can be readily packed in a uniform manner to give high porosities. Thus Davies (*5*) studied beds of fibers used for air filters over a range of $\epsilon = 0.7$ to 0.994 and obtained an empirical relationship from which the Kozeny constant can be computed. For $\epsilon < 0.8$ the equation is applicable with $k \simeq 6$, but thereafter values of k rapidly increase; at $\epsilon = 0.9$, $k = 9.7$, and at $\epsilon = 0.99$, $k = 38.8$. This is in good agreement with values in Table 1, since the orientation is neither parallel nor perpendicular to flow. Other data do not show such good agreement, but it is difficult to judge the reliability of this type of information because no exact data are reported on the fiber arrangement.

Table 1. Values of the Kozeny Constant

Fractional void volume	Flow parallel* to cylinders	Flow perpendicular to cylinders	Flow through assemblage of spheres
0.99	31.10	53.83	71.63
0.90	7.31	11.03	11.34
0.80	5.23	7.46	7.22
0.70	4.42	6.19	5.79
0.60	3.96	5.62	5.11
0.50	3.67	5.38	4.74
0.40	3.44	5.28	4.54

*For the case of flow parallel to cylinders Sparrow and Loeffler's treatment (*12a*) indicates that at high fractional void volumes arrangement does not affect permeability. At a fractional void volume of 0.5 they find $k \simeq 3.5$ for equilateral and $k \simeq 2.9$ for square array. Numerical results at lower voidages given by these authors show even greater deviations between equilateral and square arrangement.

Some data, which again show scatter, are available for flow along parallel-oriented fibers. Sullivan (*13*) has shown that if textile fibers are oriented parallel to one another and normal to the direction of flow, $k = 6$ when $\epsilon < 0.85$ and increases at higher porosities. This is in good agreement with Table 1. Sullivan (*13*) also found that with bundles of fibers combed parallel to the direction of flow, $k \simeq 2.4$ in the range of $\epsilon = 0.55$ to 0.8, somewhat lower than the values reported in Table 1. He found much lower values of k for a series of experiments with straight, smooth cylinders of various materials, though again in all these experiments there are questions of degree of randomness of arrangement, uniformity of diameter, and degree of straightness of the fibers.

Of special interest is Emersleben's (*7*) exact theoretical solution of the Navier-Stokes equations for flow parallel to uniform circular cylinders in a square array. Emersleben represented the square array of circular sections by contours of a constant value of a special Epstein zeta function (*9*), which, though it represents them closely above $\epsilon \simeq 0.8$, becomes a progressively poorer approximation at lower porosities. At $\epsilon = 0.9$, the Emersleben equation corresponds to $k = 6.3$ and compares with a value of $k = 7.3$ from Table 1; this indicates that in this range there is not much difference in the resistance between a square array and the array presumed by the free-surface model. At lower porosities agreement is poorer, but as porosity is increased, it becomes exceptionally good.

Thus for small values of ρ Emersleben (*8*) has shown that the Epstein zeta function he uses can be represented as

$$Z_0(\rho) = -\pi \ln \rho^2 - 8.234 \qquad (21)$$

For small values of ρ the Kozeny constant can also be simply represented (*6*) as

$$k = \frac{\epsilon^3\pi}{(1-\epsilon)} \times \frac{1}{Z_0(\rho)} \qquad (22)$$

For example when $\rho = 0.1$, which corresponds to $\epsilon = 0.97$, Equation (22) gives results for k about 5% too high.

Noting that $\rho^2 = (1-\epsilon)/\pi$, one finds that Equation (22) reduces to

$$k = \frac{\epsilon^3}{[1-\epsilon][-1.476-\ln(1-\epsilon)]} \qquad (23)$$

Similarly for the free-surface model, as $(1-\epsilon)^2 \ll (1-\epsilon)$, Equation (9) becomes for dilute systems

$$k = \frac{2\epsilon^3}{[1-\epsilon]\left[-3+2\ln\left(\frac{1}{1-\epsilon}\right)\right]} = \frac{\epsilon^3}{[1-\epsilon][-1.5-\ln(1-\epsilon)]} \qquad (24)$$

This excellent agreement provides validation for the free-surface model for dilute systems.

At extremely high dilutions the logarithmic term in Equation (23) will be large compared with 1.476, and as $\epsilon \rightarrow 1$, $k = -1/[(1-\epsilon)\ln(1-\epsilon)]$. Similarly for flow perpendicular to cylinders, Equation (20) reduces to $k = -2/[(1-\epsilon)\ln(1-\epsilon)]$. Thus in the extreme the resistance in the case of flow perpendicular to cylinders is just double that for axial flow. The analogous expression for flow through assemblages of spheres is $k = 1/[2(1-\epsilon)]$.

As assemblages of uniform-sized cylinders become less porous with $\epsilon < 0.3$-0.5, agreement with the values shown in Table 1 becomes poorer. With systematic rather than random arrangement agreement is also poorer. Thus for a square array of cylinders touching each other Sullivan gives an experimental value of $k = 0.83$ for axial flow. Flow perpendicular to such an array would not be possible; that is $k \rightarrow \infty$. It is possible to obtain less porous systems without having cylinders touch each other by the use of different arrangements and the employment of a range of cylinder sizes, but no data are available for flow resistance for such systems.

Another field of practical interest is the pressure drop through tube banks, such as are employed in heat exchangers. Here again it would be anticipated that the theoretical relationship developed would apply best to a nonsystematic arrangement or to one in which symmetry is maintained around each cylinder, that is the equilateral triangular spacing. One correlation quite generally used is that of Chilton and Genereaux (*4*) based on viscous flow across banks of tubes in an equilateral arrangement. Mathematically this correlation is similar to the empirical equation of Darcy and the Carman-Kozeny relationship. Thus the Darcy constant as defined above becomes

$$K = \frac{m^2[(a - D_t)/a]}{3.30} \qquad (25)$$

The present theoretical treatment gives $K = \epsilon m^2/k$, where the value of the Kozeny constant is established from Equation (20) or the values in Table 1 for flow perpendicular to cylinders. If b is taken as the longitudinal pitch (Figure 2), the fractional void volume for a staggered arrangement will be

$$\epsilon = \frac{ab - \pi D_t^2/4}{ab} \qquad (26)$$

Since $(a - D_t)/a = (ab - D_t b)/ab$, it is clear that the Chilton and Genereaux method involves the employment of b instead of $\pi D_t/4$. Since b is always bigger than $\pi D_t/4$, this method must result in a smaller constant in the denominator of the expression on the right side of Equation (25) than the Kozeny constant.

Two sets of data with transverse pitches (that is, ratio of a/D_t) of 1.25 and 1.59 were employed in the establishment of Equation (25), and thus $K = 0.061m^2$ and $0.109m^2$ respectively. For these pitches $\epsilon = 0.42$ and 0.64, corresponding to values of k from Table 1 of 5.30 and 5.85, and accordingly values of $K = 0.079m^2$ and $0.110m^2$ respectively. Thus ~~the~~ pressure drop predicted by the theoretical relationship is 23% lower than the correlation for 1.25 pitch but is in close agreement at 1.59 pitch.

More recent data by Bergelin *et al.* (*1*) indicate agreement with the theory in the low Reynolds-number range (below $N_{Re} \simeq 5$, in this case defined as $D_v G_m/\mu$) for equilateral arrangements. At higher Reynolds numbers the pressure drop is greater than predicted, but substantial deviations do not occur until Reynolds numbers over 100 are reached. Data obtained by Bergelin *et al.* with other types of tube arrangement show greater disagreement, especially with the staggered square-tube distribution. In this latter case the clearance between tubes in the direction of flow varies substantially, and so the flow pattern is not uniform around the tubes. The present theoretical method does not allow for variations in tube arrangement. Bergelin found that it was possible to bring data for each distributions into line by assuming that the pressure gradient would vary directly with the number of rows. A theoretical analysis on this basis would involve in its simplest form the treatment of resistance to flow through a single row of parallel cylinders, such as has been presented by Tamada and Fujikawa (*14*).

It may be concluded that the free-surface model presents a simple and uniform method for handling problems in viscous flow through assemblages of cylinders. It is in reasonable agreement with available data for systems involving random or uniform flow of fluids through various fibrous materials and through banks of tubes in equilateral arrangement. This technique should have further application in developing more complicated cases such as those involving simultaneous heat transfer. It should also be useful in providing suitable parameters for the correlation of experimental data.

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NOTATION

(consistent absolute units)

a = radius of cylindrical rod, transverse, pitch (Figure 1)
A, B, C, D, E, F = constants
b = radius of cylindrical rod, longitudinal pitch
dp/dx = pressure gradient
D_t = outside tube diameter
D_v = volumetric hydraulic diameter $= 4\,xm$
G_m = mass velocity through minimum cross section
k = Kozeny constant
K = Darcy constant
m = hydraulic radius, (free volume)/(exposed area)
p = pressure at any location
Q = flow rate
r = distance from axis of cylinder
u = fluid velocity in x direction, axial in case of flow parallel to cylinders, perpendicular to cylinders in case of cross flow
v_r = fluid velocity in radial direction
v_θ = fluid angular velocity
u_{avg} = average value of u
U = cylinder velocity in x direction
W = force on cylinder in x direction
x = direction of average fluid flow relative to solid cylinder
$Z_0(\rho)$ = value of the second-order Epstein zeta function with nonvanishing lower parameters along the constant-value contour which most nearly approximates the circle $\rho^2 = x^2 + y^2$

Greek Letters

∇^4 = biharmonic operator $= (\nabla^2)^2$ where ∇^2 is the Laplace operator
ϵ = fractional void volume
θ = angular distance
μ = viscosity
ρ = ratio of cylinder radius to distance between cylinder centers in a square lattice
ψ = stream function

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